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Some considerations on the Mac Dowell-Mansouri action

Andrés Anabalón

Centro de Estudios Científicos (CECS),
Casilla 1469 Valdivia, Chile
E-mail: anabalon@cecs.cl

Abstract: A precise relation is established between the Stelle-West formulation of the Mac Dowell-Mansouri approach to a gauge theory of gravity and the approach based on a gauged Wess-Zumino-Witten term. In particular, it is shown that a consistent truncation of the latter correspond to the former. A brief review of the Lovelock-Chern-Simons motivation behind the gauged WZW ones is also done.

Keywords: Gauge Symmetry, Classical Theories of Gravity.

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Contents

1. Introduction

It has been a long struggle to describe the gravitational interaction in terms of a gauge theory in the Yang-Mills sense. Up to now the results appear to be very dissimilar in odd and even dimensions. While odd dimensional Lovelock theories [1] can be used to construct gauge theories of gravity, that, moreover, have a topological interpretation using Chern-Simons (**CS**) forms [2, 3]. Even dimensional Lovelock theories appear to not be embeddable in topological structures.

However, it does exist the Mac Dowell-Mansouri [4] and Chamseddine-West [5] proposal (with the subsequent Stelle-West improvement [6]) to construct a gauge theory for (super) gravity *a la* Yang-Mills (in the sense that one of the relevant objects in the construction is a Lie algebra valued connection). This construction is elegant, somewhat reminiscent of having a topological origin and have received some considerations through the years (see for instance [7, 8]).

On the other hand it has been pointed out that due to the natural connection between Chern-Simons forms and gauged Wess-Zumino-Witten (**gWZW**) terms [9, 10] they should correspond to even dimensional gauge theories of gravity. Since the above approach and the Chamseddine-Mac Dowell-Mansouri-Stelle-West (**CMMSW**) one contains similar ingredients, namely some 0-form fields and a gauge connection, it could be expected that they should share some similarity. In fact, as is explicitly shown in this paper, they coincide when the non linear sigma model of the gWZW is accordingly restricted.

The structure of this paper is as follow: first Lovelock theories and CS theories of gravity are briefly recasted, then the gWZW structure is recalled and its properties are analyzed, the Unitary gauge is studied and implemented and finally it is shown that a consistent restriction of the gWZW theory exactly corresponds to the Stelle-West version of the CMMSW theory.

JHEP06(2008)069

2. The Lovelock series

A satisfactory description of nature is always accompanied by a reduced number of assumptions. The main difficulty to reduce the number of assumptions is that most of the times they are difficult to identify, and even after identifying them it would be far from obvious how, in a sensible way, relax them. Of course, these kind of considerations are relevant when there is at hand a theory that has been proved to be physically sensible; something that for the gravitational field, as described for the Einstein field equations

$$R_{\mu v}-\frac{1}{2}g_{\mu v}R+\Lambda g_{\mu v}=8\pi G T_{\mu v}, \tag{2.1}$$

is supported by the experimental success associated with the description of the primordial nucleosynthesis, the binary pulsar and of the solar system tests [11].

All this evidence, indeed suggests that the identification of the minimal set of assumptions that implies (2.1), is a physically relevant question. Luckily mathematicians think in uniqueness faster than physicists, and Vermeil (1917), Weyl (1922) and Cartan (1922) showed (see [1] and references therein) that it is possible to single out the l.h.s. of (2.1), in every dimension, by asking

- A rank two, symmetric tensor
- Covariant divergenceless.
- Any derivative is at most second order and the tensor is linear in them.

While the first two assumptions are motivated by what should appear at the r.h.s. of the Einstein tensor, and in fact are trivial if one begins with an action principle instead of with field equations, the third is not so. As was pointed out by Lovelock (1971) [1] it is possible to relax linearity to quasi-linearity in the second derivatives (for a discussion of quasi-linearity in this context see [12]). Remarkably, this relaxation still implies that in four dimensions the only possibility are the Einstein field equations, while, in higher dimensions, gives rise to the Lovelock series.

A nice pattern that governs the Lovelock series is given by the generalization of the relation between the Hilbert action and a two dimensional topological invariant. The Hilbert action is a non-trivial functional for the metric in all dimensions higher than two, while in two dimension it becomes a boundary term known as the Euler density. The Euler characteristic (the integral of the Euler density) is a number associated to a family of manifolds that can be related by homotopies. It exist in all dimensions, however it can be related with differentiable, geometrical features of the manifold only if it is even dimensional, in which case is given by the integral of

$$\frac{2\sqrt{|g|}}{4!4VOL(S^4)}\delta^{\mu v\lambda\rho}_{\alpha\beta\gamma\delta}R^{\alpha\gamma}{}_{\mu v}R^{\lambda\delta}{}_{\lambda\rho},\ \frac{2\sqrt{|g|}}{6!6VOL(S^6)}\delta^{\mu v\lambda\rho\eta\tau}_{\alpha\beta\gamma\delta\sigma\zeta}R^{\alpha\gamma}{}_{\mu v}R^{\lambda\delta}{}_{\lambda\rho}R^{\sigma\zeta}{}_{\eta\tau},\ \ldots \tag{2.2}$$

where the quadratic term in the curvatures correspond to the four dimensional case and the cubic to the six dimensional, the pattern in any dimension follows from the above expression.

Any term of this series is: identically zero if the number of curvatures that it contains, p, and the space-time dimension, D, is such that $2p > D$, does not contribute to the dynamics but are non trivial if $D = 2p$, and gives rise to a term of the Lovelock series if $2p < D$. This relation makes people call the terms in Lovelock series the dimensional continuation of the Euler density. Thus, the Lovelock series in dimension D contains $\left[\frac{D+1}{2}\right]$ terms, where $[\cdots]$ denotes the integer part. The terms are the dimensionally continued Euler densities of all dimensions below D, and the cosmological constant term.

Despite the condensed notation used in (2.2), is possible to note that the terms that can be added to the Lovelock Lagrangian increase its complexity with the dimension. For instance the cubic one is proportional to [13]

$$\begin{aligned} &2R^{\alpha\beta\gamma\delta}R_{\gamma\delta\lambda v}R^{\lambda v}{}_{\alpha\beta} + 8R^{\alpha\beta}{}_{\gamma\delta}R^{\gamma\lambda}{}_{\beta v}R^{\delta v}{}_{\alpha\lambda} + 24R^{\alpha\beta\gamma\delta}R_{\gamma\delta\beta v}R^{v}_{\alpha} \\ &- 3RR^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} + 24R^{\alpha\beta\gamma\delta}R_{\alpha\gamma}R_{\beta\delta} + 16R^{\alpha\beta}R_{\beta\gamma}R^{\gamma}_{\alpha} - 12RR^{\alpha\beta}R_{\alpha\beta} + R^3. \end{aligned} \tag{2.3}$$

One equation is better than one thousand words, so the previous one is enough to be convinced that a change in the notation is necessary to gain insight in the Lovelock theory. To this end is necessary to introduce the vielbein, $\bar{e}^a_\mu$, an isomorphism between the coordinate tangent space and the non-coordinate one defined by the relation $\bar{e}^a_\mu \bar{e}^b_v \eta_{ab} = g_{\mu v}$ where $\eta_{ab} = diag\,(-,+,\ldots+)$.Using this isomorphism, the curvature two form

$$R^{ab} \equiv \frac{1}{2}R^{ab}{}_{\mu v}dx^\mu \wedge dx^v \equiv \frac{1}{2}\bar{e}^a_\alpha \bar{e}^b_\beta R^{\alpha\beta}{}_{\mu v}dx^\mu \wedge dx^v,$$

and the torsion two form

$$T^a \equiv \frac{1}{2}T^a_{\mu v}dx^\mu \wedge dx^v \equiv \frac{1}{2}\bar{e}^a_\gamma T^\gamma_{\mu v}dx^\mu \wedge dx^v$$

can be defined. They are related by means of the spin connection, $\omega^{ab} \equiv \omega^{ab}_\mu dx^\mu$, through the identities

$$T^a \equiv d\bar{e}^a + \omega^a_b \wedge \bar{e}^b \equiv D\bar{e}^a, \qquad R^{ab} \equiv d\omega^{ab} + \omega^a_c \wedge \omega^{cb}, \qquad DT^a = R^{ab} \wedge \bar{e}_b. \tag{2.4}$$

Furthermore, using the convention that the wedge product ($\wedge$) is assumed between forms, the Euler characteristic can be rewritten as the integral of

$$\frac{2}{4!VOL(S^4)}\varepsilon_{abcd}R^{ab}R^{cd},\ \frac{2}{6!VOL(S^6)}\varepsilon_{abcdef}R^{ab}R^{cd}R^{ef},\ \ldots \tag{2.5}$$

With this notation and the torsionless condition, $D\bar{e}^a = 0$, the Lovelock Lagrangians in four, five, six and seven dimensions can be written as

$$\begin{aligned} \mathcal{L}_4 &= \varepsilon_{abcd}\left(\alpha_0\bar{e}^a\bar{e}^b\bar{e}^c\bar{e}^d + \alpha_1\bar{e}^a\bar{e}^b R^{cd}\right), \\ \mathcal{L}_5 &= \varepsilon_{abcde}\left(\alpha_0\bar{e}^a\bar{e}^b\bar{e}^c\bar{e}^d\bar{e}^e + \alpha_1\bar{e}^a\bar{e}^b R^{cd}\bar{e}^e + \alpha_2\bar{e}^a R^{bc}R^{de}\right), \\ \mathcal{L}_6 &= \varepsilon_{abcdef}\left(\alpha_0\bar{e}^a\bar{e}^b\bar{e}^c\bar{e}^d\bar{e}^e + \alpha_1\bar{e}^a\bar{e}^b R^{cd}\bar{e}^e + \alpha_2\bar{e}^a R^{bc}R^{de}\right)\bar{e}^f, \\ \mathcal{L}_7 &= \varepsilon_{abcdefg}\left(\alpha_0\bar{e}^a\bar{e}^b\bar{e}^c\bar{e}^d\bar{e}^e\bar{e}^f + \alpha_1\bar{e}^a\bar{e}^b R^{cd}\bar{e}^e\bar{e}^f + \alpha_2\bar{e}^a R^{bc}R^{de}\bar{e}^f + \alpha_3 R^{ab}R^{cd}R^{ef}\right)\bar{e}^g. \end{aligned}$$

JHEP06(2008)069

Where the α are dimensionful, arbitrary, coupling constants: α_0 is proportional to the cosmological constant, α_1 is related with the Newton constant while the remaining coupling constants are related to the strength of its accompanying Lovelock term. This implies that the most general Lovelock Lagrangian has $\left[\frac{D+1}{2}\right]$ coupling constants, something that would ruin any possible interpretation of it as a fundamental theory.

2.1 Chern-Simons theories

In the early eighties a related story, begun to evolve. A deep insight was being obtained on background independent field theories; since all the fundamental interactions needs the existence of a background metric to be defined, background independence was mainly associated to the requirement that the metric be a dynamical field. However, background independent field theories can also be constructed beginning with no metric at all, to my knowledge, this was pointed out to be the case by the first time with CS theories [14]. The lack of the existence of any background field implies a phase space implementation of diffeomorphism invariance, that makes the CS theories similar to General Relativity (**GR**), and it is in fact the case that, all the classical solutions of GR are contained in a CS theory [15], this highlighted the possibility of show the exact solubility of the theory [16]. Notably enough, nineteen years after these considerations, the relation between gravity and CS gravity still is matter of research and apparently is far from being completely understood [17 – 19].

The CS formulation of 2+1 GR makes the theory explicitly power counting renormalizable, this is because it can be reformulated in terms of a single gauge connection,

$$\mathcal{A} = \frac{1}{2} A_\mu^{AB} J_{AB} dx^\mu = \frac{1}{2} \omega^{ab} J_{ab} + \frac{\bar{e}^a}{l} J_{a3}, \tag{2.6}$$

where the vielbein is divided by a parameter with dimensions of length, l, in order to make the one form $\frac{\bar{e}^a}{l}$ dimensionless. The generators , J_{AB}, span the $\mathrm{SO}(2,2)$ or $\mathrm{SO}(3,1)$ algebras depending if the cosmological constant is negative or positive. The Poincaré case can be obtained by an Inönü-Wigner contraction of any of these cases.

Lets recall how the three dimensional Hilbert action can be rewritten as a CS form (For a pedagogical review see [20])[1]

$$\kappa \int_\Sigma (R - 2\Lambda) \sqrt{|g|} d^3x = \kappa \int_\Sigma \varepsilon_{abc} \bar{e}^a \left(R^{bc} \pm \frac{1}{3l^2} \bar{e}^b \bar{e}^c \right) \tag{2.7}$$

$$= \kappa l \int_\Sigma \varepsilon_{abc} e^a \left(R^{bc} \pm \frac{1}{3} e^b e^c \right) \tag{2.8}$$

$$= \kappa l \int_\Sigma \left\langle \mathcal{A} d\mathcal{A} + \frac{2}{3} \mathcal{A}^3 \right\rangle + \frac{\kappa l}{2} \int_\Sigma \varepsilon_{abc} d \left(e^a \omega^{bc} \right) \tag{2.9}$$

where in (2.7) the Palatini form of the Hilbert action is written in terms $\bar{e}^a$, ω^{ab} and $\Lambda = \mp \frac{1}{l^2}$. Note that at this point the vielbein, $\bar{e}^a_\mu$, is an invertible object that defines an isomorphism between the coordinate tangent space and the non coordinate one. In (2.8)

[1] $\langle \ldots \rangle$ stands for the invariant symmetric trace in the algebra, $\langle J_{ab} J_{c3} \rangle = \varepsilon_{abc}$.

the redefinition $\bar{e} = le$ was used. In (2.9) both objects, ω and e, are put in the same foot, making manifest the principal bundle structure of the theory.

The explicit power counting renormalizability motivated the search of a higher dimensional realization of this structure, something done in [2, 22]. CS forms exist in all odd dimensions, thus further discomposing the connection in analogy with the three dimensional case (2.6) a particular class of gravities can be found, one that contains higher powers in the curvature. It was latter realized that this gravities can be supersymmetrized, but due to the lack of the adequate superalgebras, the supersymmetrization of the CS gravities with cosmological constant was stopped at dimension seven [21]. Subsequent, exhaustive work, study most of the possible supersymmetric versions of Chern-Simons gravities [22].

Although the previous work was unrelated with the existence of Lovelock gravity it gave a hint on how to solve a fundamental problem that it has, namely the large number of, otherwise arbitrary, coupling constants present in the theory. The relative values of the $\left[\frac{D+1}{2}\right]$ coupling constants can be fixed by requiring that the local Lorentz invariance, present in any Lovelock Lagrangian when written in terms of e and ω, enlarge to anti de Sitter, de Sitter or Poincaré invariance. As was subsequently studied in [3] this enlargement of the symmetry only occurs in odd dimensions, in which case the Lovelock Lagrangian can be rewritten as a CS form.

As is discussed in [9, 10] CS theories and gWZW forms are intriscally related, thus, they define our starting point to construct a gauge theory of gravity in even dimensions.

3. Four dimensional gWZW terms

Seeking an effective lagrangian for pions it was suggested in [23] that a non-diagonally gauged version of the action principle

$$\begin{aligned} S(h,\mathcal{A}) = &-\frac{\kappa}{10}\int_{M^5}\langle h^{-1}dh(h^{-1}dh)^2(h^{-1}dh)^2\rangle \\ &+\kappa\int_{M^4}\left\langle dhh^{-1}\mathcal{A}\left(d\mathcal{A}+\frac{1}{2}\mathcal{A}^2\right)\right\rangle \\ &-\frac{\kappa}{2}\int_{M^4}\left\langle dhh^{-1}\mathcal{A}\left\{(dhh^{-1})^2+\frac{1}{2}\left[\mathcal{A}\,,dhh^{-1}\right]\right\}\right\rangle \\ &-\kappa\int_{M^4}\left\langle \mathcal{A}\mathcal{A}^h\left(\mathcal{F}+\mathcal{F}^h-\frac{1}{2}\mathcal{A}^2-\frac{1}{2}(\mathcal{A}^h)^2+\frac{1}{4}\left[\mathcal{A},\mathcal{A}^h\right]\right)\right\rangle\,, \end{aligned} \tag{3.1}$$

where

$$\mathcal{F} = d\mathcal{A}+\mathcal{A}\mathcal{A}, \qquad \mathcal{F}^h = h^{-1}\mathcal{F}h, \qquad \mathcal{A}^h = h^{-1}\mathcal{A}h + h^{-1}dh. \tag{3.2}$$

plus a kinetic term for the non-linear sigma model could represent the searched action. In our perspective, the interest in the action (3.1) is that it is diffeomorphism invariant in the same sense that Chern-Simons actions are; namely there is no necessity of a metric to define it. Thus, they are perfectly adapted to describe gravitational theories.

In order to have a gravitational interpretation of (3.1) is necessary to have a gauge group that contains the Lorentz group $so(3,1)$, and furthermore in order to have a nontrivial WZW term one is obligated to consider gauge groups that give rise to a trilinear invariant tensor. The smaller algebras that satisfy the above conditions are $so(5,1)$, $so(4,2)$,

$so(3,3)$, with generators J_{AB} and invariant tensor $\langle J_{AB}J_{CD}J_{EF}\rangle = \varepsilon_{ABCDEF}$. Along the lines discussed in the introduction, $ISO(4,1)$, can also be considered.

Let's recall some of the properties of the previous action: it is invariant under the adjoint action of the gauge group, namely

$$\mathcal{A} \to g^{-1}\mathcal{A}g + g^{-1}dg, \qquad h \to g^{-1}hg, \tag{3.3}$$

It can be noted that the action contains only even powers of h, and the invariance

$$S(h,\mathcal{A}) = S(-h,\mathcal{A}) \tag{3.4}$$

follows from this fact.

A first suggestion that this theory could make sense is given by the relation

$$S(\mathcal{A}_0,h_0) = \kappa \sinh\theta_0 \frac{3}{2}\int_{M^4} \varepsilon_{abcd}e^a e^b \left(R^{cd} + \mu e^c e^d\right) . \tag{3.5}$$

$$\mathcal{A}_0 = \frac{1}{2}\omega^{ab}J_{ab} + e^a J_{a5} \ , \quad h_0 = e^{\theta_0 J_{45}},$$

$$\mu = \frac{1}{2}\left(1 - \cosh(\theta_0)\right) \tag{3.6}$$

where (J_{ab}, J_{a5}) span the $so(3,2)$ subalgebra of $so(4,2)$. However, there is no point to have a nice construction to later mutilate it in order to obtain a desired result. Instead, if some condition is going to be imposed on the field content of an action principle, it should, at least, not modify the local symmetry present in the Lagrangian.

Parametrizing the non-linear sigma model as

$$h = \exp(\phi) = \exp\left(\frac{1}{2}J_{AB}\phi^{AB}\right), \tag{3.7}$$

and using the Killing metric $Tr(J_{AB}J_{CD}) = \eta_{AC}\eta_{BD} - \eta_{BC}\eta_{AD}$[2], the following gauge invariant condition can be imposed on the ϕ fields:

$$Tr(\phi\phi) = \frac{1}{4}\phi^{AB}\phi^{CD}\left(\eta_{AC}\eta_{BD} - \eta_{BC}\eta_{AD}\right) = \frac{1}{2}\phi^{AC}\phi_{AC} = m^2 \tag{3.8}$$

where m is a constant. Indeed, restricting the field content to the subspace defined by (3.8) do not break the symmetry of (3.1), and can be considered as a consistent restriction of it.

To further study the theory is neater to work in the Unitary gauge, something elaborated in the next section.

4. The field equations and the unitary gauge

The field equations associated with the variation with respect to h are

$$\kappa \int_{M^4} \Big\langle h^{-1}\delta h \Big\{ \left(\mathcal{F}^h\right)^2 + \mathcal{F}^2 + \mathcal{F}^h\mathcal{F} - \frac{3}{4}[\mathcal{A}^h - \mathcal{A}, \mathcal{A}^h - \mathcal{A}]\ (\mathcal{F}^h + \mathcal{F})$$
$$+ \frac{1}{8}[\mathcal{A}^h - \mathcal{A}, \mathcal{A}^h - \mathcal{A}]^2 + \frac{1}{2}(\mathcal{A}^h - \mathcal{A})[\mathcal{F}^h + \mathcal{F}, \mathcal{A}^h - \mathcal{A}]\Big) \Big\} \Big\rangle, \tag{4.1}$$

[2]Here $\eta_{AB} = (-,+,+,+,+)$.

while those associated with the connection $\mathcal{A}$ are

$$\kappa \int_{M^4} \left\langle \delta\mathcal{A} \left((\mathcal{A}^h - \mathcal{A}) \left(\mathcal{F}^h + 2\mathcal{F} - \frac{1}{4}[\mathcal{A}^h - \mathcal{A}, \mathcal{A}^h - \mathcal{A}] \right) \right) \right\rangle - (h \leftrightarrow h^{-1}) \tag{4.2}$$

4.1 A relation between the field equations

The gauge invariance of the action allows to find off-shell identities between the field equations. To see this, instead of the field variation of $(\mathcal{A}, h)$, is possible to begin with the fields $(\mathcal{A}^g,\, h^g) = (g^{-1}\mathcal{A}g + g^{-1}dg,\, g^{-1}hg)$ and consider the variational derivatives of the action with respect to $\mathcal{A}$, h and g. This process gives the same field equations for the fields ($\mathcal{A}$, h) plus an identically satisfied extra contribution.

So, with the following variations

$$\begin{aligned} \delta\left(\mathcal{A}^g\right) &= \delta g^{-1}\mathcal{A}g + g^{-1}\mathcal{A}\delta g + \delta g^{-1}dg + g^{-1}d\delta g + g^{-1}\delta\mathcal{A}g \\ &= g^{-1}\nabla\left(\delta g g^{-1}\right)g + g^{-1}\delta\mathcal{A}g, \end{aligned} \tag{4.3}$$

$$\delta h = g^{-1}\left[h, \delta g g^{-1}\right]g + g^{-1}\delta h g, \tag{4.4}$$

where $\nabla = d + [\mathcal{A},]$, three extra terms are obtained in the variational derivatives:

$$\int \left\langle g\delta g^{-1} h\mathcal{E}^h(\mathcal{A}, h)h^{-1} \right\rangle + \left\langle \delta g g^{-1}\mathcal{E}^h(\mathcal{A}, h) \right\rangle - \left\langle \delta g g^{-1}\nabla\mathcal{E}^{\mathcal{A}}(\mathcal{A}, h) \right\rangle. \tag{4.5}$$

where $\mathcal{E}^h(\mathcal{A}, h)$ are the field equations of h and $\mathcal{E}^{\mathcal{A}}(\mathcal{A}, h)$ of $\mathcal{A}$. Gauge invariance implies that this relation is identically satisfied. Thus, it follows that

$$\mathcal{E}^h(\mathcal{A}, h) - h\mathcal{E}^h(\mathcal{A}, h)h^{-1} = \nabla\mathcal{E}^{\mathcal{A}}(\mathcal{A}, h) \tag{4.6}$$

which means that the consistence condition, $\nabla\mathcal{E}^{\mathcal{A}}(\mathcal{A}, h)$, is trivially satisfied when the field equations for h, $\mathcal{E}^h(\mathcal{A}, h)$, holds. The last identity can be checked explicitly replacing the field equations at both sides of it.

4.2 A decomposition for h

An arbitrary element of a semisimple Lie algebra can be written as the adjoint action of the lie algebra on a Cartan subalgebra. So, the following local decomposition follows,

$$h = p^{-1}ap. \tag{4.7}$$

Using (4.7) the task of solving the field equations simplifies:

$$\begin{aligned} \left\langle h^{-1}\delta h\mathcal{E}^h(\mathcal{A}, h) \right\rangle &= \left\langle h^{-1}\left(\delta p^{-1}ap + p^{-1}\delta a p + p^{-1}a\delta p\right)\mathcal{E}^h(\mathcal{A}, h) \right\rangle && (4.8) \\ &= \left\langle \left(h^{-1}\delta p^{-1}ph + p^{-1}a^{-1}\delta a p + p^{-1}\delta p\right)\mathcal{E}^h(\mathcal{A}, h) \right\rangle && (4.9) \\ &= \left\langle p^{-1}\delta p\left(-h\mathcal{E}^h(\mathcal{A}, h)h^{-1} + \mathcal{E}^h(\mathcal{A}, h)\right) \right\rangle && (4.10) \\ &\qquad + \left\langle a^{-1}\delta a p\mathcal{E}^h(\mathcal{A}, h)p^{-1} \right\rangle \\ &= \left\langle a^{-1}\delta a\mathcal{E}^h(\mathcal{B}, a) \right\rangle + \left\langle p^{-1}\delta p\nabla\mathcal{E}^{\mathcal{A}}(\mathcal{A}, h) \right\rangle && (4.11) \\ &= \left\langle a^{-1}\delta a\mathcal{E}^h(\mathcal{B}, a) \right\rangle + \left\langle \delta p p^{-1}\bar{\nabla}\mathcal{E}^{\mathcal{A}}(\mathcal{B}, a) \right\rangle, && (4.12) \end{aligned}$$

where $\mathcal{B} \equiv p\mathcal{A}p^{-1} + pdp^{-1}$, $\bar{\nabla} = d + [\mathcal{B},]$ and (4.6) was used to pass from (4.10) to (4.11). On the other hand, the field equations for $\mathcal{A}$ can be rewritten as

$$\left\langle J_{AB}\mathcal{E}^{\mathcal{A}}(\mathcal{B},a)\right\rangle = 0. \tag{4.13}$$

So, p is in fact pure gauge, since it is not determined by any field equation. Thus, one is left with the milder task of solving the field equations in the so-called unitary gauge

$$\left\langle J_{AB}\mathcal{E}^{\mathcal{A}}(\mathcal{B},a)\right\rangle = 0 \qquad \left\langle C_{AB}\mathcal{E}^{h}(\mathcal{B},a)\right\rangle = 0. \tag{4.14}$$

where C_{AB} are generators along a Cartan subalgebra.

Note that the above deduction considered that all the fields of the non-linear sigma model where independently varied, while if the condition (

$$-\phi^{01}\delta\phi^{01} + \phi^{23}\delta\phi^{23} - \phi^{45}\delta\phi^{45} = 0 \tag{4.15}$$

$$\Longrightarrow \quad \delta\phi^{45} = \frac{-\phi^{01}\delta\phi^{01} + \phi^{23}\delta\phi^{23}}{\phi^{45}} \tag{4.16}$$

$$\Longrightarrow a^{-1}\delta a = \delta\phi^{01}\left(J_{01} - \frac{\phi^{01}}{\phi^{45}}J_{45}\right) + \delta\phi^{23}\left(J_{23} + \frac{\phi^{23}}{\phi^{45}}J_{45}\right) \tag{4.17}$$

Now, the main problem to obtain Einstein gravity from the gWZW term is that equations quadratic in the curvature arise when the field equations associated to the non-linear sigma model are taken in account [9, 10]. Thus, in configurations of constant ϕ, the system become overconstrained and, for instance, the unique spherically symmetric solution is flat space [24]. This quadratic constraint is proportional to a four form times ε_{abcd}, so it appears from the field equation of the form $\left\langle J_{45}\mathcal{E}^{h}(\mathcal{B},a)\right\rangle$. The restricted set of variations defined by (4.17) imply that it disappear when the ϕ field take some trivial values.

The simplest case to examine the above construction explicitly is when $ISO(4,1)$ is set as the gauge group, it is the subject of the next section.

5. The ISO(4,1) case

It is convenient to consider the $iso(4,1)$ case because it contains an abelian, invariant, subalgebra. It allows to restrict the non-linear sigma model to take its values on this subalgebra without affecting the local symmetry of the action. Decomposing the $iso(4,1)$ algebra in its $so(3,1)$ irreducible parts the generators reads (J_{ab}, P_c, T_c, W), where (J_{ab}, P_c) span the $iso(3,1)$ subalgebra and (J_{ab}, T_c) span the $so(4,1)$ subalgebra. The commutation relations are

$$[J_{ab}, J_{cd}] = -J_{ac}\eta_{bd} + J_{bc}\eta_{ad} - J_{bd}\eta_{ac} + J_{ad}\eta_{bc}, \tag{5.1}$$

$$[J_{ab}, T_c] = -T_b\eta_{ac} + T_a\eta_{bc},$$

$$[J_{ab}, P_c] = -P_b\eta_{ac} + P_a\eta_{bc}, \tag{5.2}$$

$$[T_a, P_c] = -W\eta_{ac},$$

$$[T_a, W] = P_a,$$

$$[T_a, T_b] = -J_{ab}. \tag{5.3}$$

$$a = 0, \ldots, 3 \qquad \eta_{ab} = (-,+,+,+,) \tag{5.4}$$

and, correspondingly, the connection is written as

$$\mathcal{A} = \frac{1}{2}\omega^{ab}J_{ab} + c^a P_a + b^a T_a + \Phi W \ , \tag{5.5}$$

while the curvature reads

$$\mathcal{F} = \frac{1}{2}\left(R^{ab} - b^a b^b\right) J_{ab} + \left(db^a + \omega^{ac}b_c\right) T_a + \left(dc^a + \omega^{ab}c_b + b^a\Phi\right) P_a + \left(d\Phi - b^a c_a\right) W. \tag{5.6}$$

The simplest thing that one can do is to consider that the non-linear sigma takes its values along the generators (P, W):

$$h = \exp(z^A P_A), \qquad P_A = (P_a, W), \tag{5.7}$$

In this way the gWZW action takes the simple form

$$S(h, \mathcal{A}) = 3\kappa \int_{M^4} z^A \varepsilon_{ABCDE} \Omega^{BC} \Omega^{DE} \ , \tag{5.8}$$

$$\Omega = \frac{1}{2}\Omega^{AB}J_{AB} = \frac{1}{2}\left(R^{ab} - b^a b^b\right) J_{ab} + \left(db^a + \omega^{ac}b_c\right) T_a, \tag{5.9}$$

which after imposing the gauge invariant constraint $z^A z_A = m^2$, gives rise to standard Einstein gravity. In the above action part of the original $ISO(4,1)$ symmetry is realized in a trivial way and the remanent symmetry is just $\mathrm{SO}(4,1)$.

Thus, we have exactly reproduced de CMMSW gauge theory of gravity. Too much exactly; the main two drawbacks of the this theory are still present [7]. That is the necessity to impose the gauge invariant constraint (3.8) by hand and the lack of a good reason to consider a sector of the gauge connection to be invertible (the vielbein). Interestingly enough the second of these issues is solved by a relation that looks exactly like a term of (4.1) (see equation 14 in [7]), something that would deserve further consideration.

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JHEP06(2008)069